Seat No. : _______________

ZM-140

May-2014

M.Sc. Sem.-II

407 : Mathematics

(Differential Geometry – I)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Find parametrizations of the following level curves : **7**

(i) $x^2 + y^2 = 1$

(ii) $\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$

OR

Let $r(t) = (t - \sin t, 4 - \cos t), 0 < t < 2\pi$

Calculate the arc length of this curve.

(b) Answer briefly any **two :** **4**

(i) Let $r(t) = (e^t \cos t, e^t \sin t), -\infty < t < \infty$. Find the angle between r(t) and the tangent vector at r(t).

(ii) Let $r(t) = (t, -\sinh(t)), -\infty < t < \infty$. Is r regular ?

(iii) Suppose that the regular curve r(t) lies on the unit sphere. What is the angle between r(t) and the tangent vector at r(t) ?

(c) Answer very briefly all parts. **3**

(i) Sketch the curve $y - 3x = 1$

(ii) Sketch the curve $y - x^2 = 0$

(iii) Sketch the curve $x^2 + y^2 = 1$

2. (a) Compute $k, \tau, \bar{t}, \bar{n}$ and $\bar{b}$ and verify that the Frenet-Serret equations are satisfied for the curve **7**

$$r(t) = \left(\frac{1}{3}(2-t)^{3/2}, \frac{1}{3}t^{3/2}, \frac{2-t}{\sqrt{2}}\right), 0 < t < 2$$

OR

Compute $k, \tau, \bar{t}, \bar{n}$ and $\bar{b}$ and verify that the Frenet-Serret equations are satisfied for the curve

$$r(t) = \left(\frac{-3}{5}\sin t, 1 + \cos t, \frac{4}{5}\sin t\right), 0 < t < 2\pi$$

(b) Answer briefly any **two :** **4**

(i) Let $\gamma(s)$ and $\delta(s)$ be two unit speed curves in $\mathbb{R}^3$ with the same curvature $k(s) > 0$ and the same torsion $\tau(s)$ for all s. How are the two curves related ?

(ii) Compute the torsion of the circular helix

$$r(\theta) = (2 \cos \theta, 2 \sin \theta, 3\theta)$$

(iii) Let r(s) be a unit speed plane curve in $\mathbb{R}^2$. Define the signed curvature of r.

(c) Answer very briefly all parts : **3**

(i) Let r(t) be a regular curve in $\mathbb{R}^3$. Write down (without proof) a formula for its curvature.

(ii) Give an example (without proof) of a unit speed curves in $\mathbb{R}^2$ whose signed curvature is – 1 at every point.

(iii) Define the centre of curvature $\varepsilon(s)$ of r at the point r(s), where r has nowhere zero signed curvature.

3. (a) Show that if $f(x, y)$ is a smooth function, then its graph $\{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}$ is a smooth surface with atlast consisting of a single regular surface patch **7**

$$\sigma(u, v) = (u, v, f(u, v))$$

OR

Show that the ellipsoid $\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{4^2} = 1$ is a smooth surface.

(b) Answer briefly any **two :** **4**

(i) Define the tangent space at a point P of a surface S. Give an example of a non-zero tangent vector at (0, 0, 1) on the sphere $x^2 + y^2 + z^2 = 1$.

(ii) Give a basis for the tangent plane to the surface patch $\sigma(u, v) = (u, v, u^2 + v^2)$ at the point (1, 1, 2).

(iii) Show that the unit sphere cannot be covered by a single surface patch.

(c) Answer very briefly all parts : **3**

(i) Give (without proof) a parametrization $\sigma(u, v)$ of the unit sphere $x^2 + y^2 + z^2 = 1$.

(ii) Give (without proof) a parametrization $\sigma(u, v)$ of the plane $x + y + z = 0$.

(iii) Name a surface in $\mathbb{R}^3$ which is not orientable.

4. (a) Define a generalized cylinder S. Give a parametrization $\sigma(u, v)$ for S. Find necessary conditions for σ to be regular. **7**

Define a generalized cone S. give a parametrization $\sigma(u, v)$ for S. Find necessary conditions for σ to be regular.

OR

Define a ruled surface S. give a parametrization $\sigma(u, v)$ for S. Find necessary conditions for σ to be regular.

Define a surface of revolution. Give (without proof) a parametrization $\sigma(u, v)$ for the torus obtained by rotating a circle of radius 2 in the xz-plane centered at $\sigma(3, 0, 0)$, around the z-axis.

(b) Answer briefly any **two :** **4**

(i) Show that the quadric $x^2 + 2y^2 + 6x - 4y + 3z = 7$ is an elliptic paraboloid.

(ii) Show that if a quadric contains three points on a straight line, it contains the whole line.

(iii) Let P be a point which lies on the sphere $x^2 + y^2 + z^2 = 1$ and also on the plane $z = 0$. Show that the tangent planes of these surfaces at P are perpendicular to each other.

(c) Answer very briefly all parts. **3**

(i) Write down (without proof) a parametrization for the hyperbolic paraboloid $z = x^2 - y^2$.

(ii) Define a triply orthogonal system (of surfaces).

(iii) Give an example of a triply orthogonal system. (Do not prove)

5. (a) Consider the surface patches : **7**

$\sigma(u, v) = (\cosh(u) \cos(v), \cosh (u) \sin v, u)$,

$-\infty < u < \infty, 0 < v < 2\pi$, and

$\tau(u, v) =(u \cos (v), u \sin (v), v)$,

$-\infty < u < \infty, 0 < v < 2\pi$

Show that the map that takes $\sigma(u, v)$ to

$\tau(\sinh (u), v)$ is an isometry.

OR

Show that Mercator's parametrization of the sphere

$\sigma(u, v) = (\text{sech} (u) \cos (v), \text{sech}(u) \sin(v), \tanh (u))$,

$-\infty < u < \infty, 0 < v < 2\pi$,

is conformal.

(b) Answer briefly any **two :** **4**

(i) Calculate the first fundamental form of the surface $\sigma(u, v) = (u - v, u + v, u^2 + v^2)$.

(ii) Let $S \subset \mathbb{R}^3$ be the plane z =0. Is the map $f : S \to S$ given by

$f(x, y, 0) = (x + y, x - y, 0)$

an isometry ?

(iii) Let $S \subset \mathbb{R}^3$ be the plane z = 0. Is the map $f : S \to S$ given by

$f(x, y, 0) = (2x, 3y, 0)$

conformal ?

(c) Answer very briefly all parts. **3**

(i) Define an isometry $f : S_1 \to S_2$.

(ii) Define a conformal map $f : S_1 \to S_2$.

(iii) Define an equiareal map $f : S_1 \to S_2$.